METHODS NOTES
KYLA ORCHARD

$\log_a(1) = 0$
$\log_a(a) = 1$

$\ln(4x - 2) = -1$
$4x - 1 = e^{-1}$ $4x = 2 + 1/e$

LOGS

$a^x = y \rightarrow \log_a(y)$
$2^3 = 8 \rightarrow \log_2 8 = 3$

$\log_a(x) = \log_b(x) \div \log_b(a)$

$e^x = y \rightarrow x$
$= \log_e(y) = \ln y$

$\log_a(x) + \log_a(y)$
$= \log_a(x \times y)$
$\log_a(x) - \log_a(x)$
$= \log_a(x \div y)$
$\log_a(x)^n = n\log_a(x)$
$\log_a\left(\frac{1}{x}\right) = -\log_a(x)$

$\lim_{n \to 0}\left(1 + \frac{1}{n}\right)^n = e$

$\lim_{n \to 0}\left(1 - \frac{1}{n}\right)^n = 1/e$

$\log_a(a^x) = x$
$a^{\log_a x} = x$

$\frac{\log 27}{\log 3^2} = \frac{\log 27}{2\log 3}$
$= \frac{\log 3^3}{2\log 3} = \frac{3\log 3}{2\log 3}$
$= 3/2 = 1.5$

$3\log_2 6 - \log_2 27$
$= \log_2 216 - \log_2 27$
$= \log_2(216/27)$
$= \log_2 8$
$= 3$

LOG FUNCTIONS

flip sign of b

$y = \log_a(x - b) + c$
asymptote $x = b$
x-int $(a^{-c} + b, 0)$
point $(a^{1-c} + b, 1)$
$-b$ = right $+c$ up
b = left $-c$ down

DIFFERENTIATION

$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$

chain ($y = [f(x)]^n$) $n[f(x)]^{n-1} \times f'(x)$

product $f'(x)g(x) + g'(x)f(x)$

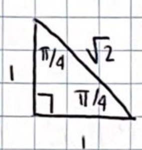
quotient $\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

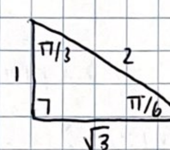
y	y'
$y = ax^n$	$y' = n \times ax^{n-1}$
$y = e^{f(x)}$	$f'(x) \times e^{f(x)}$
$y = 1/x = x^{-1}$	$-1/x^2 = -x^{-2}$
$y = \pm\sin x$	$\pm\cos x$
$y = \pm\cos x$	$\mp\sin x$
$y = \ln[f(x)]$	$f'(x)/f(x)$
$y = a^x$	$\ln a \times a^x$

INVERSE FUNCTIONS

1. rearrange func. to make x the subject instead of y
2. swap variables x & y $= f^{-1}(x)$

$y = a^x$ $\log_a y = x\log_a a$
$y = \log_a a^x$ $\log_a y = x$
$\therefore y = \log_a x$





TURNING POINTS

	$f'(x)$	$f''(x)$
min	0	+
max	0	−
horiz poi	0	0
vert poi	+ or −	0

SKETCHING FUNCTIONS

1. $x + y$ ints
2. stationary pts $f'(x)$ $f''(x) = 0$
3. nature $f''(x)$
4. $x \rightarrow \pm\infty$

INCREMENTS

$\delta y \approx \frac{dy}{dx} \times \delta x$

$\frac{\delta y}{y} \approx \frac{dy}{dx} \times \frac{\delta x}{x}$ ← % change

$x = 3 \rightarrow 2.98$
$y = 3x^2 - 2x$
$\delta y \approx dy/dx \times \delta x$
$\approx (6(3) - 2) \times (-0.02)$
≈ -0.32 $= \downarrow$ by 0.32

GROWTH + DECAY

$A = A_0 e^{kt}$

$dA/dt = kA_0 e^{kt} = kA$

1000 bacteria decay in min after 7am
1/2 size = 7mins
$A = 0.5A_0$
$0.5 = e^{7k}$ $k = \ln 0.5/7$
$\therefore 0.5A_0 = A_0 e^{7k}$
$\ln(0.5) = 7k$
$k = -0.99$

OPTIMISATION

1. diagram + variables
2. reduce variables → 2
3. dy/dx of $f(x)$
4. $dy/dx = 0$ solve for t.p's
5. nature of t.p's d^2y/dx^2
6. optimal dimensions for min or max

SKETCHING DERIVATIVES

- local min/max are x-int of $f'(x)$
- func ↑, $f'(x)$ is above x-axis + vice versa
- poi, $f'(x) \rightarrow$ t.p

INTEGRATION

$\int x\,dx \quad \int x\,dx = \frac{x^2}{2} + c$

$\int x^n\,dx$	$\frac{x^{n+1}}{n+1} + c$
$\int f'(x)[f(x)]^n dx$	$\frac{[f(x)]^{n+1}}{n+1} + c$
$\int e^{f(x)}\,dx$	$\frac{e^{f(x)}}{f'(x)} + c$
$\int \sin(x)\,dx$	$-\cos(x) + c$
$\int \cos(x)\,dx$	$\sin(x) + c$

$\int_a^b f(x) = -\int_b^a f(x)$

$\int a \times f(x)\,dx = a \times \int f(x)\,dx$

$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int_b^a f(x)\,dx + \int_c^b f(x)\,dx = \int_c^a f(x)\,dx$

AREA UNDER A CURVE

1. roots $y=0$
2. integrals above/below x axis

AREA BETWEEN CURVES

$\int_a^b$ upper − lower dx

$\int_c^d$ right − left dx

FUNDAMENTAL THEOREM

$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

$\int_a^b f(x)\,dx$

$= F(b) - F(a)$

$\int_0^x \ln(t)\,dt \quad \ln(x)$

PROBABILITY

$P(\bar{A}) = 1 - P(A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

RANDOM VARIABLES

$E(X) = \mu$

$var(X) = \sigma^2$

$\sqrt{var(X)} = \sigma$

$var(X) = E(X^2) - [E(X)]^2$

$Y = aX + b$

$E(Y) = aE(X) + b$

$var(Y) = a^2 var(X)$

$\int e^{-6x} + 2\sqrt{x} - 4\pi\,dx$

$2\sqrt{x} = 2x^{1/2}$

$\int f(x)\,dx = \frac{-e^{-6x}}{6} + \frac{4x^{3/2}}{3} - 4\pi x + c$

$\int 2\sin(4-3x)$

$= \frac{-2\cos(4-3x)}{-3} = \frac{2\cos(4-3x)}{3} + c$

$\int 3^{2x}$

$\int e^{2\ln 3x} = \frac{e^{2\ln 3x}}{2\ln(3)} = \frac{3^{2x}}{2\ln 3} + c$

$\int_0^{\pi/4} (2x - \cos x)$

$= [x^2 - \sin x]_0^{\pi/4} = \left[\left(\frac{\pi}{4}\right)^2 - \sin\left(\frac{\pi}{4}\right) - 0\right] = \frac{\pi^2}{16} - \frac{\sqrt{2}}{2}$

$\int \frac{1-12x^2}{3x}$

$= \int \frac{1}{3x} - 4x\,dx = \frac{1}{3}\ln x - \frac{4x^2}{2}$

$= \frac{\ln x}{3} - 2x^2 + c$

RECTILINEAR

displacement

velocity

acceleration

v = point on line

s = area under

a = gradient

total distance travelled

$= \int_a^b |v(t)|\,dt$

DRV's

$\sum P(X=x) = 1$

$0 \le P(X=x) \le 1$

$P(X < a) = P(X \le a-1)$

$P(X > a) = P(X \ge a+1)$

-probs + → 1

-no negative probs

$E(X) = \sum x\,p(x)$

$Var(X) = \sum (x-\mu)^2 p(x)$

BERNOULLI

-success vs failure

-independent

$X \sim Ber(p)$ ← success prob.

$var(X) = \sigma^2 = p(1-p)$

$E(X) = p$

$S.D = \sqrt{p(1-p)}$

BINOMIAL

-multiple bernoulli

$X \sim Bin(n, p)$ success, ↑ trials

$P(X=x) = \binom{n}{x}(p)^x(1-p)^{n-x}$

${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r!)r!}$

$E(X) = np$

$var(X) = np(1-p)$

$S.D = \sqrt{np(1-p)}$

$P(X) =$ binomPDF(x, n, p)

$P(A \le X \le B) =$ binomCDF(A, B, n, p)

$P(X \le k) =$ invBinCDF$(P(X \le k), n, p)$

10 Q's 4 options, guessing

$X \sim Bin(10, 0.25)$

$P(5 \le X \le 10) = 0.0781$

0.1 chance out of 6 = ?

$= 0.9^5 \times 0.1$

$= 0.0590$

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BINOMIAL CONT.

roll a 5 at least 2 times on a dice from 10 throws

$X \sim B(10, 0.167)$

$P(X \geq 2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - 0.1609 - 0.3225$

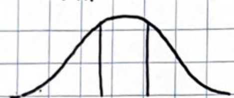
$= 0.5166$

NORMAL DISTRIBUTION

↑ prob closer to mean

$X \sim N(\mu, \sigma^2)$

mean ↗ ↖ variance



1 s.d = 68%

2 s.d = 95%

3 s.d = 99.7%

a% of data lies below a^{th} percentile

$P(X < K_a) = a$

$P(A \leq X \leq B) =$ normCDF(A, B, σ, μ)

find K given $P(X \leq K)$

invnormCDF(tail setting, $P(X \leq K)$, σ, μ)

find value of 67^{th} percentile

$P(X < K) = 0.67$

K = invnormCDF(left, 0.67, 5, 20)

$= 22.20$

CONFIDENCE + ERROR MARGINS

$$\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = (CI_L, CI_U)$$

99% = 2.58 ← z score

95% = 1.96

90% = 1.645

$Z_{c\%} = -1 \times$ invNormCDF("L", c%, 1, 0)

ERROR MARGIN

- 1/2 width of CI

$$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad E \propto z \qquad E \propto \frac{1}{\sqrt{n}}$$

$$CI = \hat{p} \pm E \qquad p = \frac{CI_L + CI_U}{2} \qquad E = \frac{CI_U + CI_L}{2}$$

CRV's

- decimals

$\int p(x)\,dx = 1$

$0 \leq P(X=x) \leq 1$

$P(X<a) = P(X \leq a)$

$P(X>a) = P(X \geq a)$

$E(X) = \int x\,p(x)\,dx$

$var(X) = \int (x-\mu)^2\,p(x)\,dx$

UNIFORM DISTRIBUTION

- constant probability

$X \sim U(a, b)$

upper ↗ ↑ lower

$f(x) = \frac{1}{b-a}$

area = 1

$E(X) = \frac{1}{2}(a+b)$

$var(X) = \frac{1}{12}(b-a)^2 \qquad s.d = \sqrt{\frac{1}{12}(b-a)^2}$

$P(c \leq X \leq d)$

$\int_c^d \frac{1}{b-a}\,dx$

find k given $P(X \leq k)$

$\int_a^k \frac{1}{b-a}\,dx = P(X \leq k)$

Z SCORES

$\mu = 0$, s.d = 1

- number of s.d away

$Z \sim (0, 1^2)$

$z = \frac{x-\mu}{\sigma}$

RANDOM SAMPLING

- selection bias
  - undercoverage
  - nonresponse
  - voluntary response
- response bias
  - leading question
  - loaded question

↓ sampling error:

- ↑ sample size
- every n^{th} person
- stratified: sample groups reflecting size

CENTRAL LIMIT THEOREM

random variable X

μ = pop. mean $\bar{x}$ = sample mean

mean = $\bar{x}$ s.d. = $\sigma/\sqrt{n}$

z-score = $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$

bernoulli   p = pop mean $\hat{p}$ = sample mean

$\mu = \hat{p}$

$s.d. = \sqrt{\frac{p(1-p)}{n}}$

$z = \frac{\hat{p}-p}{\sqrt{p(1-p)/n}}$

18% are overcooked, 150 pizzas find dist'n

$\mu = \hat{p} = 0.18 \qquad \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.18(1-0.18)}{150}} = 0.03$

changing CI

1. CI bounds → p $p = (CI_L + CI_U)/2$
2. margin of E $E = CI_U - p$
   $E = p + CI_L$
3. $E_{new} = z_{new}/z_{old} \times E_{old}$
4. new CI $= p \pm E_{new}$

90% CI = (0.38, 0.45)
determine 95%

$p = \frac{0.38 + 0.45}{2} = 0.415$

$E = 0.45 - 0.415 = 0.035$

$E_{new} = \frac{1.96}{1.645} \times 0.035$

95% CI $= 0.415 \pm 0.0417$
$= (0.3733, 0.4567)$

30/150 like chocolate
find 95% CI

$p = \frac{30}{150} = 0.2 \quad \sigma = \sqrt{\frac{p(1-p)}{n}}$
$= 0.0327$

$z = 1.96$
95% CI $= (0.2 \pm 1.96 \times 0.0327)$
95% confident that pop. propor. lies between 0.1359 & 0.2641 13.59 → 26.41% hate chocolate

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if $f'(x) = e^{-2x}$ find $f(x)$ given $f(0) = -2$

$f(x) = -\frac{1}{2} e^{-2x} + c$
$-2 = -1/2 + c$
$c = -3/2$
$f(x) = -\frac{e^{-2x}}{2} - 3/2$

$d/dx \ x \ln x$
$= \ln x + 1$

$\int \frac{d}{dx}(x \ln(x))\, dx$
$= \int \ln x + 1 \ dx$
$x \ln x = \int \ln x + x + c$
$\therefore \int \ln x = x \ln x - x + c$

$C = 4e^{-0.05t}$
$dc/dt = -0.05(4)e^{-0.05t}$

$f(x) = ax^2(x-2)$ for $0 \le x \le 2$

$a\int_0^2 x^3 - 2x^2 dx = 1$

$\left[\frac{x^4}{4} - \frac{2x^3}{3}\right]_0^2 = 1$

$a[4 - 16/3] = 1$
$a = -3/4$

↳ $P(X \geq 1.2)$

$-3/4 \int_{1.2}^2 x^3 - 2x^2 dx$

$= 0.5248$

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how many times ↑ is margin of E of sample 1225 vs 11025

$E \propto \frac{1}{\sqrt{1225}} = \frac{1}{35}$
$= 1/105 \ \therefore 3$

90% CI = (0.12, 0.46)

$p = \frac{0.12 + 0.46}{2} = 0.29$

$E = \frac{0.46 - 0.12}{2} = 0.17$

$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
$= 0.17$
$= 1.645\sqrt{\frac{0.29(1-0.29)}{n}}$
$n = 20 \quad \therefore 0.29 \times 20$
$= 6$ students

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maximise profit:

$P(x) = \frac{50\ln(\frac{x}{2})}{x^2} \quad 1.5 \le x \le 10$

$P'(x) = \frac{x^2\, 50 \frac{2}{x}\frac{1}{2} - 50\ln(\frac{x}{2})2x}{x^4}$

$= \frac{50x - 100x\ln(\frac{x}{2})}{x^4}$

$= \frac{50 - 100\ln(\frac{x}{2})}{x^3}$

$= 0$ when $50/100 = \ln(x/2)$
$1/2 = \ln(x/2)$
$e^{1/2} = x/2$
$\therefore x = 2e^{1/2}$

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find median

$-\frac{3}{4}\int_0^m x^3 - 2x^2 dx = \frac{1}{2}$

$\int_0^m x^3 - 2x^2 dx = -2/3$

$\frac{m^4}{4} - \frac{2m^3}{3} = -2/3 \qquad 3m^4 - 8m^3 = -8$
$m = 1.228\ldots$

$v = 2\sin(t/3 + \pi/6)$
$x(t) = 2\int \sin(t/3 + \pi/6)$
$= -2 \cdot 3\cos(t/3 + \pi/6) + c$
$0 = -6\cos\pi/6 + c$
$-6^3\sqrt{3}/2 + c$
$c = 3\sqrt{3}$
$\therefore x(t) = -6\cos(t/3 + \pi/6) + 3\sqrt{3}$

$\hat{p} = 0.5$
$\rightarrow E = \sqrt{\hat{p}(1-\hat{p})}$

$\frac{dE}{d\hat{p}} = \frac{1}{2}(\hat{p} - \hat{p}^2)^{-1/2}(1 - 2\hat{p})$
$= 0$

when $\hat{p} = 1/2$

$\left.\frac{d^2E}{d\hat{p}^2}\right|_{\hat{p}=0.5} = -2$ max

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95% CI (0.342, 0.558)
$\hat{p} - E \le p \le \hat{p} + E$
$\hat{p} - E = \ldots$
$\hat{p} + E = \ldots$

$\therefore \hat{p} = \frac{\ldots}{2}$

$= 0.45$
$E = 0.558 - 0.45$
$= 0.108$
$\therefore 0.108 = 2.576\sqrt{\frac{0.45(1-\ldots}{n}}$
$n = 141$
# of vehicles
$= 141 \times 0.45$
$= 63$